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The Tunneling Radiation Characteristics of Kerr-Newman de Sitter Black Hole

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The strictly thermal spectrum in dragging coordinate system and the tunneling radiation characteristics of stationary axisymmetry Kerr-Newman de Sitter black hole is studied. The result shows that the tunneling rates at the event and cosmological horizon are related to the change of Bekenstein-Hawking entropy and that the factual radiation spectrum is not strictly pure thermal. Thus an exact correction to the Hawking thermal spectrum is present.

KEY WORDS: Kerr-Newman de Sitter black hole; tunneling rate; energy conservation; angular momentum conservation; Bekenstein-Hawking entropy.

1. INTRODUCTION

In 1975, Hawking proved theoretically that the thermal radiation could be happened from black holes, and the corresponding temperature is true (Hawking, 1975). In 1976, Damour and Ruffini applied not second quantilization but Relativity Quantum Mechanics in curved space- time to verify the Hawking radiation from black holes (Damour and Ruffini, 1976). In 1988, Sannan advanced the method by using the idea of Quantum Field Theory and Quantum Statistics (Sannan, 1988). From then on, a series of research on that of stationary and non-stationary black holes has been carried out (Jiang *et al.*, 2005; Liu and Zhao, 2001; Xu, 1998; Yang and Lin, 2001; Zhao *et al.*, 1999). However, all of the derived results are the precisely thermal because of the common fixed background space-time. Following that, a paradox of information loss is present with the black hole evaporation, which means that the pure quantum state will be disintegrated to the mixture, using the language of Quantum Field Theory, the ingoing state is the pure state, but the outgoing is the mixture, thus the underlying unitary theory is disobeyed. In addition, Hawking regarded that the thermal radiation of black hole is the contribution

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of the quantum tunneling effect triggered by the vacuum fluctuation near the event horizon, namely, a pair of particles creates just inside the horizon, the positive energy particle is tunneling out and the negative anti-particle is absorbed by black hole. In other words, we can consider that the particles created just outside the horizon, the negative energy anti-particle is tunneled into the horizon because the negative energy orbit is only existed into the horizon, thus the positive energy particle is left outside the horizon and moves towards the infinite distance and form Hawking thermal spectrum. Both the two narrative styles have a tunneling process, so the tunneling barrier should be found to truly describe the tunneling process and obtain the true radiate spectrum. But till now, the causes of the tunneling barrier are unclear for us. The related references do not use the language of quantum tunneling method to discuss Hawking radiation, so strictly speaking, it is not the quantum tunneling method.

Recently, a method to describe Hawking radiation as tunneling process, where a particle moves in dynamic geometry, has been developed by Kraus and Wilczek and elaborated upon by Parikh and Wilczek who carried out research on the tunneling radiation characteristics of static spherically symmetric Schwarzchild black hole and Reissner-Nordström black hole (Parikh, 2000, 2002, 2004). The results display that the derived radiation spectrum is not strictly thermal under the consideration of energy conservation and the unfixed space-time background, which provides a correct amendment to Hawking radiation spectrum. The method overcomes the defects of Hawking radiation, and points out that the tunneling barrier was offered by self-gravitation among particles. Following this method, Hemming and Keski-Vakkuri have investigated the Hawking radiation from Anti-de Sitter black holes (Hemming and Keski-Vakkuri, 2001), and Medved has studied those from a de Sitter cosmological horizon (Medved, 2002). But all of those are limited to that of the spherically symmetric black holes. In this paper, we extend the Parikh's work to study the Hawking radiation as tunneling from stationary axi-symmetry Kerr-Newman de Sitter black holes, and obtain the Hawking spectrum in dragging coordinate system and the tunneling rates at the event and cosmological horizon. In Section 2, the horizons and the infinite red-shift surface are researched. In Section 3, we obtain the precisely thermal spectrum in dragging coordinate system. The main work is left in Section 4, and investigates the tunneling radiation characteristics of the black hole. Finally, Section 5 contains discussion and conclusion.

2. THE EVENT HORIZON AND THE INFINITE RED-SHIFT SURFACE

The space-time line element of Kerr-Newman de Sitter black hole can be written as (Dehgham and KhajehAzad, 2003)

$$ds^2 = -\frac{\Delta_r^2}{\rho^2}\left(dt_{kNS} - \frac{a}{\Xi}\sin^2\theta d\varphi\right)^2 + \frac{\rho^2}{\Delta_r^2}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt_{kNS} - \frac{(r^2+a^2)}{\Xi}d\varphi\right)^2, \qquad (1)$$

where t_{kNS} is the coordinate time of the black hole, and

$$\Delta_r^2 = (r^2 + a^2)\left(1 - \frac{r^2}{l^2}\right) - 2Mr + q^2, \quad \Delta_\theta = 1 + \frac{a^2}{l^2}\cos^2\theta,$$
$$\Xi = 1 + \frac{a^2}{l^2}, \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{2}$$

According to the null surface equation $g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0$, we have

$$r^4 + (a^2 - l^2)r^2 + 2Ml^2 r - (a^2 + q^2)l^2 = 0, \tag{3}$$

which is the horizon equation of the black hole, and there are a negative root r_- and three roots r_0, r_h, r_c corresponding to the inner, outer and cosmological horizon of the black hole respectively, namely

$$\begin{aligned} r_0 &= -\sqrt{Z_1} + \sqrt{Z_2} + \sqrt{Z_3}, \\ r_h &= \sqrt{Z_1} - \sqrt{Z_2} + \sqrt{Z_3}, \\ r_c &= \sqrt{Z_1} + \sqrt{Z_2} - \sqrt{Z_3}, \end{aligned} \tag{4}$$

in which

$$\sqrt{Z_1} = \sqrt{\frac{l^2 - a^2}{6}}\left[1 + \sqrt{1 - \frac{12l^2(q^2 + a^2)}{(l^2 - a^2)^2}}\cos\frac{\alpha}{3}\right]^{1/2},$$

$$\sqrt{Z_2} = \sqrt{\frac{l^2 - a^2}{6}}\left[1 - \sqrt{1 - \frac{12l^2(q^2 + a^2)}{(l^2 - a^2)^2}}\cos\left(\frac{\alpha}{3} + \frac{\pi}{3}\right)\right]^{1/2},$$

$$\sqrt{Z_3} = \sqrt{\frac{l^2 - a^2}{6}}\left[1 - \sqrt{1 - \frac{12l^2(q^2 + a^2)}{(l^2 - a^2)^2}}\cos\left(\frac{\alpha}{3} - \frac{\pi}{3}\right)\right]^{1/2},$$

$$\alpha = \arccos\left\{-\frac{(l^2 - a^2)[(l^2 - a^2)^2 + 36l^2(q^2 + a^2)] - 54M^2l^4}{[(l^2 - a^2)^2 - 12l^2(q^2 + a^2)]^{3/2}}\right\}.$$

Now, Let's calculate the horizon area of the black hole, under the condition of constant-time slice and $r = r_h$, the line element (1) will be reduced to

$$d\sigma^2 = \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\sin^2\theta}{\rho^2\,\Xi^2}\left[\Delta_\theta\left(r_h^2 + a^2\right)\right]d\varphi^2, \tag{5}$$

so we have

$$g = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} = \frac{\sin^2\theta}{\Xi^2}\left(r_h^2 + a^2\right)^2, \tag{6}$$

and then the event horizon of the black hole is

$$A_h = \int dA' = \int \sqrt{g} d\theta d\varphi = \frac{4\pi}{\Xi} \left(r_h^2 + a^2 \right). \tag{7}$$

the cosmological horizon is

$$A_c = \frac{4\pi}{\Xi} \left(r_c^2 + a^2 \right). \tag{8}$$

From $g_{00} = 0$, the infinite red-shift surface equation can be written as

$$\Delta_r^2 - \Delta_\theta \sin^2\theta a^2 = 0, \tag{9}$$

obviously, the infinite red-shift surface and the event horizon of the black hole are not coincident with each other. So performing the dragging coordinate transformation as

$$\dot{\varphi} = \frac{d\varphi}{dt_{KNS}} = -\frac{g_{03}}{g_{33}} = \Omega, \tag{10}$$

we have

$$ds^2 = \hat{g}_{00} dt_{KNS}^2 + \frac{\rho^2}{\Delta_r^2} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2, \tag{11}$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = -\frac{\Delta_\theta \Delta_r^2 \rho^2}{\Delta_\theta (r^2+a^2)^2 - \Delta_r^2 a^2 \sin^2\theta}$. At this momentum, when $\hat{g}_{00} = 0$, the infinite red-shift surface is coincident with the horizons of the black hole in the dragging coordinate system.

3. THE HAWKING PRECISELY THERMAL SPECTRUM IN DRAGGING COORDINATE SYSTEM

In this section, we will discuss the thermal radiation spectrum of uncharged particles. In the curved space-time, Klein-Gordon equation of uncharged particles can be expressed as

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} \cdot g^{\mu\upsilon} \frac{\partial}{\partial x^\upsilon} \Phi \right) = \mu^2 \Phi. \tag{12}$$

Substituting Eq. (11) into Eq. (12) yields

$$\hat{g}^{00} \frac{\partial^2 \Phi}{\partial t_k^2} + g^{11} \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{\sqrt{-g}} \frac{\partial \Phi}{\partial r} \frac{\partial}{\partial r} (\sqrt{-g} g^{11}) + g^{22} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{\sqrt{-g}} \frac{\partial \Phi}{\partial \theta} \frac{\partial}{\partial \theta} (\sqrt{-g} g^{22}) = \mu^2 \Phi. \tag{13}$$

Carrying on the separation variable to Eq. (13) as

$$\Phi = e^{-i\omega t_{KNS}} R(r) \psi(\theta) e^{im\varphi}, \tag{14}$$

and considering the effect of the dragging coordinate transformation, we can obtain the following expression

$$\frac{d^2R(r)}{dr^2} + \frac{1}{g^{11}}\left(\frac{g^{11}}{\sqrt{-g}}\frac{\partial}{\partial r}\sqrt{-g} + \frac{\partial g^{11}}{\partial r}\right)\frac{dR}{dr} + \frac{1}{g^{11}}\frac{R(r)}{\psi(\theta)}\{G(r,\theta)\}$$
$$= \frac{1}{g^{11}}\left[\mu^2 + \left(\omega + m\frac{g_{03}}{g_{33}}\right)^2 \hat{g}^{00}\right]R(r) \tag{15}$$

where $G(r,\theta) = \frac{\Delta_\theta}{\rho^2}\frac{d^2\psi(\theta)}{d\theta^2} + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial\theta}(\sqrt{-g}g^{22})\frac{d\psi(\theta)}{d\theta}$. And then, introducing the tortoise coordinate transformation

$$r_* = \frac{1}{2\kappa_h} In(r - r_h), \tag{16}$$

we have

$$\frac{d^2R(r)}{dr_*^2} - 2\kappa_h\frac{dR}{dr_*} + 2\kappa_h(r - r_h)\left(\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial r} + \frac{1}{g^{11}}\frac{\partial g^{11}}{\partial r}\right)\frac{dR}{dr_*}$$
$$+ \frac{4\kappa_h^2(r - r_h)^2}{g^{11}}\frac{R(r)}{\psi(\theta)}\{G(r,\theta)\}$$
$$= \frac{4\kappa_h^2(r - r_h)^2}{g^{11}}\left[\mu^2 + \left(\omega + m\frac{g_{03}}{g_{33}}\right)^2 \hat{g}^{00}\right]R(r), \tag{17}$$

where $\kappa_h = \frac{1}{2l^2(r^2+a^2)}(r_h - r_-)(r_h - r_0)(r_c - r_h)$ is the surface gravity of the event horizon. In the vicinity of the event horizon, namely $r \to r_h$, we have

$$\frac{4\kappa_h^2(r - r_h)^2}{g^{11}}\left[\mu^2 + \left(\omega + m\frac{g_{03}}{g_{33}}\right)^2 \hat{g}^{00}\right]R(r) = -(\omega - m\Omega_H)^2 R(r). \tag{18}$$

Substituting Eq. (18) into Eq. (17), we can get the standard wave equation near the black hole

$$\frac{d^2R(r)}{dr_*^2} + (\omega - \omega_0)^2 R(r) = 0. \tag{19}$$

where $\omega_0 = m\Omega_H = \frac{ma\Xi}{r_h'^2 + a^2}$. Solving Eq. (19) can we obtain the radial wave function of uncharged particles ingoing and outgoing the black hole as

$$\psi_{in} = e^{-i\omega v}, \quad \psi_{out} = e^{-i\omega v}e^{2i(\omega-\omega_0)r_*}. \tag{20}$$

where $v = t_{KNS} + \frac{\omega-\omega_0}{\omega}r_*$ is the advanced Eddington-Finkelstein coordinate. ψ_{out} can be written as follows near the event horizon

$$\psi_{out} = e^{-i\omega v}(r - r_h)^{i(\omega-\omega_0)/\kappa}. \tag{21}$$

Obviously, ψ_{out} has a logarithm singularity. By analytical continuation rotating $-\pi$ through the lower-half complex $r-$plane as

$$(r \to r_h) \to |r - r_h|\, e^{-i\pi} = (r_h - r)\, e^{-i\pi}, \tag{22}$$

and using the Damour-Ruffini stretch method of analysis, and extending it to the inside of the event horizon, we can get the spectrum of the Hawking radiation

$$N_\omega = \frac{1}{e^{\frac{(\omega-\omega_0)}{T}} - 1} = \frac{1}{e^{\alpha_h A_h} - 1}, \tag{23}$$

where

$$T = \frac{\kappa_h}{2\pi}, \quad \alpha_h = \frac{\Xi(\omega - \omega_0)l^2}{2\left[-2(a^2 + q^2)l^2 r_h^{-1} + 3Ml^2 + (a^2 - l^2)r_h\right]}, \tag{24}$$

where A_h is the area of the event horizon of the black hole. Obviously, the derived Hawking radiation spectrum at the event horizon is related to the fixed area of the event horizon A_h.

In the same word, the Hawking radiation spectrum at the cosmological horizon is

$$N_\omega = \frac{1}{e^{\frac{(\omega-\omega_0)}{T}} - 1} = \frac{1}{e^{\alpha_c A_c} - 1}, \tag{25}$$

where

$$T = \frac{\kappa_c}{2\pi}, \quad \alpha_c = \frac{\Xi(\omega - \omega_0)l^2}{2\left[-2(a^2 + q^2)r_c^{-1}l^2 + 3Ml^2 + (a^2 - l^2)r_c\right]}. \tag{26}$$

Obviously, the derived Hawking radiation spectrum at the cosmological horizon is still related to the fixed area of the cosmological horizon A_c.

In summary, the thermal prosperity of Kerr-Newman de Sitter black hole can be described in the dragging coordinate system, the expressions (23) and (25) are based on the fixed background space-time. In fact, the horizons change with the black hole evaporation, and the factual background space-time are dynamical.

4. THE GENERAL PAINLEVE COORDINATE TRANSFORMATION AND THE QUANTUM EFFECT VIA TUNNELING

Although the infinite red-shift surface and the horizons are coincident with each other in the dragging coordinate system, there still exists a coordinate singularity at the horizon of the black hole, which brings us inconvenience to investigate the tunneling behavior across the horizon of the black hole. So the new coordinate transformation is needed. Parikh applied the Painlevé coordinate $dt_S = dt + f'(r)\,dr$ to eliminate the coordinate singularity and investigate the

Hawking radiation of the sphere-symmetric black hole via tunneling. In this paper, the space-time is axi-symmetrical, so we should go on performing the general Painlevé coordinate transformation as (Jiang and Wu, 2005; Yang, 2005; Yang *et al.*, 2006; Zhang and Zhao, 2005)

$$dt_{KNS} = dt + F(r, \theta) dr + G(r, \theta) d\theta, \tag{27}$$

where $F(r, \theta)$ and $G(r, \theta)$ are two functions of r and θ, and the integrability condition of Eq. (27) satisfies $\partial_\theta F(r, \theta) = \partial_r G(r, \theta)$. Substituting Eq. (27) into Eq. (11) and ordering the derived constant-time slice of the spacetime flat Euclidean in radial, so the line element of the black hole in the general Painlevé coordinate system is

$$\begin{aligned} ds^2 = {} & \hat{g}_{00} dt^2 + dr^2 \pm 2\sqrt{\hat{g}_{00}(1 - g_{11})} dt dr + [\hat{g}_{00} G^2(r, \theta) + g_{22}] d\theta^2 \\ & + 2\sqrt{\hat{g}_{00}(1 - g_{11})} G(r, \theta) dr d\theta + 2\hat{g}_{00} G(r, \theta) dt\, d\theta. \end{aligned} \tag{28}$$

where the positive sign (+) denotes the space-time of the outgoing particle, and the negative sign (−) represents the line element of the ingoing particle. Substituting Eq. (28) into the Landau's condition of the coordinate clock synchronization, we can also obtain $\partial_\theta F(r, \theta) = \partial_r G(r, \theta)$. Thus the Painlevé-Kerr-Newman de Sitter line element satisfies the Landau's condition of the coordinate clock synchronization, apart from this, there are many other superior features: The infinite red-shift surface and the horizons of the black hole are coincident with each other; The metric is regular at the the horizons of the black hole; the measure on the surfaces of constant-time slices is the same as that of flat spacetime. All of these characters are advantageous for us to study the Hawking radiation via tunneling.

Considering the uncharged particle's radial motion and tunneling from the event horizon as an ellipsoid shell, the particle should be still an ellipsoid shell during the tunneling process to conserve the symmetry of the space-time. So from Eq. (28), the radial null geodesics equation are given as

$$\dot{r} = \frac{dr}{dt} = -\sqrt{\hat{g}_{00}(1 - g_{11})} \pm \sqrt{-\hat{g}_{00} g_{11}} = \frac{\pm\rho^2\sqrt{\Delta_\theta} - \sqrt{\rho^2 \Delta_\theta \left(\rho^2 - \Delta_r^2\right)}}{\sqrt{\Delta_\theta (r^2 + a^2)^2 - \Delta_r^2 a^2 \sin^2\theta}}. \tag{29}$$

where the plus (minus) sign denotes the outgoing (ingoing) geodesics.

Now, Let's move on to discuss the tunneling radiation characteristics of the black hole. For the sake of simplicity, we only consider the tunneling radiation of uncharged particles. In our discussion, we can consider the picture of a pair of virtual particles spontaneously created just inside the horizon, the positive energy virtual particle can tunnel out and the negative energy anti-particle is absorbed by the black hole. Taking the particle's self-gravitation interaction, energy conservation, angular momentum conservation into account, and fixing the total mass and angular momentum of the space-time and allowing those of the black hole to fluctuate, when the particle is tunneled out as an ellipsoid shell of energy ω and angular momentum ωa, then the mass and the angular momentum of the black hole will be replaced by $(M-\omega)$ and $(M-\omega)\,a$ respectively. Meanwhile the event horizon will shrink, we refer to the cases pre- and post shrinking as two turning points of potential barrier, the distance between the two turning points is the width of potential barrier and decided by the energy of outgoing particle. At this critical moment, Eqs. (4), (28) and (29) will be modified by replacing the mass parameter M with $(M-\omega)$, and the shell of energy will travel on the modified geodesics

$$\dot{r} = \frac{dr}{dt} = \frac{\pm\rho^2\sqrt{\Delta_\theta} - \sqrt{\rho^2\Delta_\theta\left(\rho^2 - \Delta_r'^2\right)}}{\sqrt{\Delta_\theta(r^2+a^2)^2 - \Delta_r'^2 a^2\sin^2\theta}}, \tag{30}$$

where $\Delta_r'^2 = (r^2+a^2)(1-\frac{r^2}{l^2}) - 2(M-\omega)r + q^2$. In the WKB approximation, the emission rate from a radiating source can be expressed in terms of the imaginary part of the action for an outgoing positive energy particle as

$$\Gamma \sim e^{-2\mathrm{Im}S}. \tag{31}$$

In the dragging coordinate system, the coordinate φ does not appear in the line element expression. That is, φ is an ignored coordinate in the Lagrangian function. To eliminate this freedom completely, the action should be written as

$$\begin{aligned}\mathrm{Im}S &= \mathrm{Im}\int_{t_i}^{t_f}(L - P_\varphi\dot{\varphi})dt = \mathrm{Im}\left[\int_{r_i}^{r_f} P_r dr - \int_{\varphi_i}^{\varphi_f} P_\varphi d\varphi\right] \\ &= \mathrm{Im}\left[\int_{r_i}^{r_f}\int_0^{p_r} dP_r' - \int_{\varphi_i}^{\varphi_f}\int_0^{p_\varphi} dP_\varphi' d\varphi\right]dr.\end{aligned} \tag{32}$$

where r_i and r_f are just inside and outside the barrier at the event horizon through which the particle is tunneling. Applying Hamilton's equations

$$\begin{aligned}\dot{r} &= \left.\frac{dH}{dP_r}\right|_{(r;\varphi,P_\varphi)}, \\ \dot{\varphi} &= \left.\frac{dH}{dP_\varphi}\right|_{(\varphi;r,P_r)}, \quad dH_{(\varphi;r,P_r)} = \Omega' dJ\end{aligned} \tag{33}$$

where $H = \frac{M}{\Xi^2}$ is the total energy of the black hole, and when the particle of energy ω emits out of the event horizon, then $H = \frac{M-\omega}{\Xi^2}$ $dH = -\frac{1}{\Xi^2}d\omega$, $P_\varphi = J$. Substituting Eq. (33) into Eq. (32) can we obtain

$$\mathrm{Im}S = \mathrm{Im}\int_{\frac{M}{\Xi^2}}^{\frac{M-\omega}{\Xi^2}}\int_{r_i}^{r_f}\left(\frac{dH'}{\dot{r}} - \frac{\Omega' dJ'}{\dot{r}}\right)dr = \mathrm{Im}\int_0^{\omega}\int_{r_i}^{r_f} -\frac{1}{\Xi^2}\left(\frac{d\omega'}{\dot{r}} - \frac{a\Omega' d\omega'}{\dot{r}}\right)dr$$

$$= \mathrm{Im} \int_0^{\omega} \int_{r_i}^{r_f}$$

$$- \frac{1}{\Xi^2} \frac{\sqrt{\Delta_\theta (r^2 + a^2)^2 - \Delta_r'^2 a^2 \sin^2\theta}[\rho^2 \sqrt{\Delta_\theta} + \sqrt{\rho^2 \Delta_\theta (\rho^2 - \tilde{\Delta}_r'^2)}]}{\rho^2 \Delta_\theta \tilde{\Delta}_r'^2}$$

$$\times t(1 - a\Omega')d\omega' dr \tag{34}$$

where $\tilde{\Delta}_r'^2 = (r^2 + a^2)(1 - \frac{r^2}{l^2}) - 2(M - \omega')r + q^2$. Equation (34) tells us that there is a pole at the event horizon of the black hole after the particle emission. The integral can be evaluated by deforming the contour around the pole, so as to ensure that positive energy solution decay in time. So we have

$$\mathrm{Im} S = \mathrm{Im} \int_{r_i}^{r_f} -\frac{\pi r i}{\Xi} dr = -\frac{\pi}{2\Xi} \left(r_f^2 - r_i^2 \right). \tag{35}$$

Substituting Eq. (35) into Eq. (31), we can get the tunneling rate at the event horizon as

$$\Gamma \sim \mathrm{e}^{-2\mathrm{Im} S} = \mathrm{e}^{\frac{\pi}{\Xi}(r_f^2 - r_i^2)} = \mathrm{e}^{\frac{A_h' - A_h}{4}} = \mathrm{e}^{\Delta S_{BH}}. \tag{36}$$

where ΔS_{BH} is the Bekenstein-Hawking entropy at the event horizon, A_h and A_h' are the area of the event horizon before and after the particle of energy ω emission. Comparing Eq. (36) with Eq. (23), we can learn that the tunneling rate at the event horizon provides a correct modification to Hawking radiation spectrum.

Now, we will discuss the Hawking radiation of the particle via tunneling at the cosmological horizon. Different from the particle's tunneling behavior of the event horizon, the particle is found tunneled into the cosmological horizon. So when the particle with energy ω tunnels into the cosmological horizon, Eqs. (4), (28) and (29) will be modified by replacing the mass parameter M with $(M + \omega)$ after taking the self-gravitation action into account. Therefore, when the particle with energy ω tunnels into the cosmological horizon, the null radial geodesics can be written as

$$\dot{r} = \frac{dr}{dt} = \frac{\pm \rho^2 \sqrt{\Delta_\theta} - \sqrt{\rho^2 \Delta_\theta (\rho^2 - \Delta_r''^2)}}{\sqrt{\Delta_\theta (r^2 + a^2)^2 - \Delta_r''^2 a^2 \sin^2\theta}}, \tag{37}$$

where $\Delta_r'^2 = (r^2 + a^2)(1 - \frac{r^2}{l^2}) - 2(M + \omega)r + q^2$. Different from the event horizon, $H = -\frac{M}{\Xi^2}$ and $H' = -\frac{M+\omega}{\Xi^2}$ are the total energy of the black hole at the cosmological horizon before and after the particle of energy ω tunneling into, and then the imaginary part of the action at the cosmological horizon can be expressed as

$$\mathrm{Im} S = \mathrm{Im} \int_{-\frac{m}{\Xi^2}}^{-\frac{m-\omega}{\Xi^2}} \int_{r_{ic}}^{r_{fc}} \left(\frac{dH'}{\dot{r}} - \frac{\Omega' dJ'}{\dot{r}} \right) dr$$

$$= \mathrm{Im} \int_0^{\omega} \int_{r_{ic}}^{r_{fc}} -\frac{1}{\Xi^2} \left(\frac{d\omega'}{\dot{r}} - \frac{a\Omega' d\omega'}{\dot{r}} \right) dr$$

$$= \text{Im} \int_0^{\omega} \int_{r_i}^{r_f}$$

$$- \frac{1}{\Xi^2} \frac{\sqrt{\Delta_\theta (r^2 + a^2)^2 - \Delta_r'^2 a^2 \sin^2\theta}[\rho^2 \sqrt{\Delta_\theta} + \sqrt{\rho^2 \Delta_\theta (\rho^2 - \tilde{\Delta}_r''^2)}]}{\rho^2 \Delta_\theta \tilde{\Delta}_r''^2}$$

$$\times (1 - a\Omega') d\omega' dr, \tag{38}$$

where r_{ic} and r_{fc} are the locations of the cosmological horizon before and after the particle of energy ω tunneling into, and $\tilde{\Delta}_r''^2 = (r^2 + a^2)(1 - \frac{r^2}{l^2}) - 2(M + \omega') r + q^2$. Obviously, there exists a pole at the cosmological horizon. The integral can be evaluated by deforming the contour around the pole. Doing the ω' integral firstly yields

$$\text{Im}S = \text{Im} \int_{r_{ic}}^{r_{fc}} -\frac{r\pi i}{\Xi} dr = -\frac{\pi}{2\Xi} \left(r_{fc}^2 - r_{ic}^2 \right), \tag{39}$$

So the tunneling rate at the cosmological horizon is

$$\Gamma \sim e^{-2\text{Im}S} = e^{\frac{\pi}{\Xi}(r_{fc}^2 - r_{ic}^2)} = e^{\frac{A'_c - A_c}{4}} = e^{\Delta S_{CH}}, \tag{40}$$

where $\Delta S_{CH} = S_{CH}(M + \omega) - S_{BH}(M)$ is the Bekenstein-Hawking entropy of the cosmological horizon, and A_c and A'_c are the areas of the cosmological horizon before and after the particle of energy ω tunneling into. Comparing Eq. (40) with Eq. (25), we can learn that the tunneling rate at the cosmological horizon still provides a correct modification to Hawking radiation spectrum.

5. DISCUSSION AND CONCLUSION

In special cases, when $l \to \infty$, $a = 0$, the line element (1) will reduced to Reissner-Nordström black hole where the event horizons before and after the particle with energy ω emission are $r_h^R = m + \sqrt{m^2 - q^2}$, $r_h'^R = (m - \omega) + \sqrt{(m - \omega)^2 - q^2}$, and the corresponding areas are $A_h^R = 4\pi (r_h^R)^2$ and $A_h'^R = 4\pi (r_h'^R)^2$ respectively. From Eq. (23), we can obtain the precisely thermal spectrum of the black hole as

$$N_\omega = \frac{1}{e^{\omega/T} - 1} = \frac{1}{e^{\alpha_h^R A_h^R} - 1}, \tag{41}$$

where $\alpha_h^R = \frac{-\omega}{2\sqrt{m^2 - q^2}}$. Substituting the Bekenstein-Hawking entropies of the black hole before and after the particle emission

$$S_{BH} = \frac{1}{4} A_h^R = \pi (2m^2 - q^2 - 2m\sqrt{m^2 - q^2}). \tag{42}$$

$$S'_{BH} = \frac{1}{4} A'^R_h = \pi (2 (m - \omega)^2 - q^2 - 2 (m - \omega) \sqrt{(m - \omega)^2 - q^2}), \tag{43}$$

into Eq. (36), the tunneling rate of the black hole is derived as

$$\Gamma \sim e^{\frac{A'^R_h - A^R_h}{4}} = e^{S'_{BH} - S_{BH}} = e^{-2\pi [2\omega(m - \frac{\omega}{2}) + m\sqrt{m^2 - q^2} - (m - \omega)\sqrt{(m-\omega)^2 - q^2}]} = e^{\Delta S_{BH}}. \tag{44}$$

The expression (44) supports the Parikh's results.

The existence of the Hawking tunneling characteristics in Kerr-Newman de Sitter black hole causes the radiation at the horizon of the black hole. When a particle with energy ω is radiated, the horizon of the black hole will changed after taking energy conservation and angular momentum conservation into consideration, and then the background geometry should be dynamical. So the factual radiation spectrum will be deviated from the strictly thermal one. Kerr-Newman de Sitter black hole is a general axi-symmetry black hole, and the tunneling effect can be happen at the cosmological and event horizon. Equations (36) and (40) provide a correct modification to the corresponding Hawking radiation spectrum, where $(r_f^2 - r_i^2)$ and $(r_{fc}^2 - r_{ic}^2)$ are connected with the energy ω carried by the outgoing and ingoing particle. In special cases, from the expression (44) we can learn that the tunneling rate is related to higher exponent of ω, only neglecting the items of higher exponent of ω can we obtain the same result as Eq. (41). Obviously, the Hawking thermal radiation spectrum is strictly thermal but the factual radiation spectrum is not, which provides a meaningful correction.

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